Chaotic properties of dilute two- and three-dimensional random Lorentz gases. II. Open systems

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(Received 21 August 2000; published 27 December 2000)

We calculate the spectrum of Lyapunov exponents for a point particle moving in a random array of fixed hard disk or hard sphere scatterers, i.e., the disordered Lorentz gas, in a generic nonequilibrium situation. In a large system which is finite in at least some directions, and with absorbing boundary conditions, the moving particle escapes the system with probability one. However, there is a set of zero Lebesgue measure of initial phase points for the moving particle, such that escape never occurs. Typically, this set of points forms a fractal repeller, and the Lyapunov spectrum is calculated here for trajectories on this repeller. For this calculation, we need the solution of the recently introduced extended Boltzmann equation for the nonequilibrium distribution of the radius of curvature matrix and the solution of the standard Boltzmann equation. The escape-rate formalism then gives an explicit result for the Kolmogorov Sinai entropy on the repeller.

DOI: 10.1103/PhysRevE.63.016312 PACS number(s): 05.20.Dd, 05.40.−a, 05.45.−a, 05.60.−k

I. INTRODUCTION

In this paper we extend the analysis of the chaotic properties of dilute, random Lorentz gases given in Ref. [1] [denoted by (I)] to include open systems with absorbing boundaries. The Lorentz gas consists of a point particle moving in a system of identical hard disk ($d=2$) or hard sphere ($d=3$) scatterers of radius a. In a dilute, random Lorentz gas, the average distance between the scatterers is large compared to their radius, a, and the scatterers are placed at random on the plane or in space without overlapping each other. The interest in open systems, with absorbing boundaries, is occasioned by the escape-rate method of Gaspard and Nicolis [2] which relates the coefficient of diffusion for the moving particle in the Lorentz gas to the dynamical properties of particles on the set of trajectories that never escape from the system.

The method of Gaspard and Nicolis is based on an identity in the theory of open, hyperbolic dynamical systems, called the escape-rate formula. This formula is an expression for the rate of decay of the probability, $P(t)$, of finding a moving particle in an open region, V, surrounded by absorbing boundaries, at time t. If the motion of the moving particle is hyperbolic, the time dependence of $P(t)$ is exponential, decaying as $\exp(-\gamma t)$, where the escape-rate, γ, is given by

$$\gamma=\sum_{\lambda_i>0}\lambda_i(\mathcal{R})-h_{KS}(\mathcal{R}). \tag{1}$$

Here $\mathcal{R}$ denotes the set of initial phase points for the moving particle which are on trajectories that never escape from the system. This set of points is called a ''repeller'' in the phase space, typically of measure zero with respect to the usual Lebesgue measure, with unstable and stable manifolds characterized by Lyapunov exponents $\lambda_i(\mathcal{R})$, and Kolmogorov–Sinai (KS) entropy $h_{KS}(\mathcal{R})$. The sum in Eq. (1) is only over the positive Lyapunov exponents, and the Lyapunov exponents and KS entropy are to be calculated with respect to an appropriate measure on the repeller [3,4]. Mathematically rigorous proofs of the escape-rate formula, under a variety of circumstances, have been given by several authors [5–7], although the application of interest here, to large, random Lorentz gas systems with open boundaries, still needs a rigorous proof.

Equation (1) may be considered as the microscopic expression for the escape-rate of the particle from the open region, V. A macroscopic expression for the escape-rate is provided by the diffusion equation satisfied by $P(t)$ on large time and large space scales,

$$\frac{\partial P(t)}{\partial t}=D\nabla^2 P(t), \tag{2}$$

where D is the macroscopic diffusion coefficient for the moving particle in V. The solution of this equation, for long times, and for absorbing boundary conditions is of the form

$$P(t)=\exp\left[-\left(\frac{c}{L^2}\right)Dt\right], \tag{3}$$

where L is a length characterizing the distance to the absorbing boundary of interior points in V, and c is a numerical factor determined by the shape of V and the absorbing boundary conditions. Since the microscopic and the macroscopic expressions for the escape-rate describe the same escape process, they have to be identical, which leads to the Gaspard–Nicolis formula

$$D=\lim_{L\to\infty}\frac{L^2}{c}\left[\sum_{\lambda_i>0}\lambda_i(\mathcal{R})-h_{KS}(\mathcal{R})\right]. \tag{4}$$

Here the limit $L\to\infty$ is taken to remove finite size corrections, including the effects of microscopic boundary layers. It is worth mentioning that the Green–Kubo microscopic expressions for the diffusion coefficient, D, and Sinai's expression for the sum of positive Lyapunov exponents represent these quantities as infinite time integrals over appropriate dynamical quantities. In order to apply Eqs. (2)–(4), without having to worry about subtleties due to the possible slow convergence of these integrals (due to long-time-tail effects) we assume that all of the dynamical quantities, λ_i, h_{KS} reach their long-time asymptotic values on a time scale which is short compared to the hydrodynamic time scale on which the diffusion equation applies. (This assumption is certainly a reasonable one for the low density cases we consider here.) The escape-rate formula for diffusion, Eq. (4), and the generalizations to other transport coefficients [8], show a striking connection between the macroscopic quantities that control hydrodynamic processes, the transport coefficients, and the microscopic quantities that describe the chaotic dynamics taking place on the repeller. A detailed discussion of this can be found elsewhere [9,10].

The purpose of this paper is to provide an analytical calculation of the positive Lyapunov exponents on the repeller, $\lambda_i(\mathcal{R})$, for the random, dilute Lorentz gas, and to use these, together with known values for the diffusion coefficient, D, to determine $h_{KS}(\mathcal{R})$, the KS entropy of the trajectories on the repeller. We take the same approach as in I where we used kinetic theory arguments to calculate the Lyapunov spectra of two and three dimensional random, dilute Lorentz gases in equilibrium. Here we are concerned with a nonequilibrium situation where particles escape from the system. We will see that spatial inhomogeneities introduced by the absorbing boundaries will require some significant modifications to our previous calculations. There we used mean-free-path arguments and some results from the theory of products of random matrices to determine low density values for the individual Lyapunov exponents and we used an extended Lorentz–Boltzmann equation as an efficient method for determining the sum of the positive Lyapunov exponents. Here we will do the same for the Lyapunov exponents on the repeller. We mention that before we developed this method [11] analytical results for Lyapunov exponents on repellers had only been obtained for simple one dimensional models [12,13]. Otherwise one had to use numerical methods [14].

It will be helpful to recall some ideas from I. There we obtained the individual Lyapunov exponents as well as their sums in terms of various averages over functions of a radius of curvature (ROC) matrix $\boldsymbol{\rho}$, using an appropriate distribution function. A central notion introduced in I is the use of an extended Lorentz–Boltzmann equation (ELBE) to determine the distribution of the elements of the radius of curvature matrices needed for our calculations. The ELBE was derived heuristically in I. In the absence of external fields acting on the moving particle it is given by

$$\frac{\partial F}{\partial t}+\vec{v}\cdot\nabla F+\dot{\boldsymbol{\rho}}:\frac{\partial F}{\partial\boldsymbol{\rho}}$$
$$=na^{d-1}\int d\hat{n}|\vec{v}\cdot\hat{n}|\left[\Theta(\vec{v}\cdot\hat{n})\int d\boldsymbol{\rho}'\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'(\boldsymbol{\rho}))\right.$$
$$\left.\times F(\vec{r},\vec{v}',\boldsymbol{\rho}',t)-\Theta(-\vec{v}\cdot\hat{n})F(\vec{r},\vec{v},\boldsymbol{\rho},t)\right]. \tag{5}$$

The notation is exactly the same as in I. The primed variables denote the restituting values, which lead to the unprimed values after a collision, and $\hat{n}$ denotes a unit vector from the center of a scatterer to the point of impact of the moving particle at a collision. The solution of Eq. (5) is normalized according to

$$\int d\boldsymbol{\rho}'F(\vec{r},\vec{v},\boldsymbol{\rho}',t)=f_B(\vec{r},\vec{v},t), \tag{6}$$

where f_B is the solution of the standard Lorentz-Boltzmann equation [16]. In this paper we will assume, as in I, that the elements of the inverse of the restituting matrix, $[\boldsymbol{\rho}']^{-1}$, are typically small compared to the inverse of the scatterer radius a^{-1}. As a result, we may simplify the delta function appearing in the collision integral on the right-hand side of Eq. (5).

The plan of the paper is as follows: In Sec. II we discuss the problem of calculating averages on the repeller in phase space. There we argue that it is necessary to introduce a survival probability for a particle that is inside the system at time t, to be still inside the system at time $t+T$. We argue that this probability is needed to guarantee that the properties we calculate are actually those of the repeller and not merely some asymptotic properties of a set of particles that eventually escape from the region A. (Not taking this survival probability into account led to an erroneous result in a previous publication on this subject, subsequently corrected in an erratum [11].) We will set up the formalism that will allow us to calculate the Lyapunov spectrum on the repeller in Sec. III. In Sec. IV, we will treat the two dimensional case, and in Sec. V, the three dimensional case. We conclude in Sec. VI with a summary of our results, a number of remarks, and a discussion of some interesting open questions.

II. AVERAGING ON THE REPELLER

To treat the open system correctly it is necessary to develop a tool which guarantees that the quantities we calculate are the actual Lyapunov exponents for trajectories on the repeller. In general, the measure on a repeller is a very singular object. For large systems we expect that this measure will be very similar for all typical configurations of scatterers, when observed on length scales small compared to typical macroscopic length scales (system size) but large compared to the mean free path between collisions. Therefore averages on the repeller may be replaced by averages over smooth reduced distribution functions for the light particle alone obtained by averaging over all configurations of scatterers. Further support of this picture is provided by the observation that for large systems the fractal dimension is close to the embedding dimension, as follows from Young's formula [13] relating the Lyapunov exponents to the information dimension of the repeller.

For dilute Lorentz gases without escape, the probability density of the ROC matrix elements is obtained as the time independent solution of the generalized Lorentz–Boltzmann equation for $F(\boldsymbol{\rho})$ given by Eq. (5). For a system with escape, the solution of the time dependent Lorentz–Boltzmann equation only determines the probability of finding a particle at point $\vec{r}$ with velocity $\vec{v}$ with a ROC matrix $\boldsymbol{\rho}$ at time t. It does not contain any information about the future behavior of this particle. More specifically it does not exclude the possibility of this particle leaving the system at a later time $t+\tau$. To obtain the smoothed density *on the repeller* we therefore have to weight this Lorentz–Boltzmann density with the survival probability. This is the conditional probability $S(\vec{r},\vec{v},t|t+T)$, that a particle at point $\vec{r}$ with velocity $\vec{v}$ at time t will still be in the system at the time $t+T$. It does not depend on $\boldsymbol{\rho}$ because all trajectories in a bundle are infinitesimally close to each other. The introduction of the survival probability in order to obtain a proper description of the repeller is very reminiscent of, and essentially identical to, methods used to compute fractal dimensions and other properties of attractors and repellers in more traditional dynamical systems calculations [13,15].

The survival probability that we need, $S(\vec{r},\vec{v},t|t+T)$, can, in d dimensions, be written as an integral over the conditional probability $S(\vec{r},\vec{v},t|\vec{r}',\vec{v}',t+T)$ as

$$S(\vec{r},\vec{v},t|t+T)=\int d^d\vec{r}'d^d\vec{v}'S(\vec{r},\vec{v},t|\vec{r}',\vec{v}',t+T), \quad (7)$$

with initial condition

$$S(\vec{r},\vec{v},t|\vec{r}',\vec{v}',t)=\delta(\vec{r}-\vec{r}')\,\delta(\vec{v}-\vec{v}'). \quad (8)$$

The average of a function $g(\boldsymbol{\rho})$ on the repeller is then given by

$$\langle g(\boldsymbol{\rho})\rangle_{\text{Rep}} = \lim_{t\to\infty}\lim_{T\to\infty}\frac{\int d\boldsymbol{\rho}d\vec{r}d\vec{v}F(\vec{r},\vec{v},\boldsymbol{\rho},t)S(\vec{r},\vec{v},t|t+T)g(\boldsymbol{\rho})}{\int d\boldsymbol{\rho}d\vec{r}d\vec{v}F(\vec{r},\vec{v},\boldsymbol{\rho},t)S(\vec{r},\vec{v},t|t+T)}. \quad (9)$$

The limit $T\to\infty$ has to be taken first, to guarantee that only the trajectories which never leave the system are counted. The limits of numerator and denominator vanish separately, since the repeller is a set of Lebesgue measure zero. The conditional probability $S(\vec{r},\vec{v},t|\vec{r}',\vec{v}',t+T)$ is the solution of the ordinary Lorentz–Boltzmann equation for open systems $f_S(\vec{r}',\vec{v}',t+T)$ with the initial condition specified in Eq. (8). Due to time reversibility and time translation invariance, this probability is the same as that for finding a particle at point $\vec{r}$ with velocity $-\vec{v}$ at time t, that was at point $\vec{r}'$ with velocity $-\vec{v}'$ at time $t-T$. Thus,

$$\begin{aligned}S(\vec{r},\vec{v},t|\vec{r}',\vec{v}',t+T)&=S(\vec{r}',-\vec{v}',t-T|\vec{r},-\vec{v},t)\\&=S(\vec{r}',-\vec{v}',0|\vec{r},-\vec{v},T).\end{aligned} \quad (10)$$

The integration in Eq. (7) with respect to $\vec{r}'$ and $\vec{v}'$ allows us to replace $S(\vec{r},\vec{v},t|t+T)$ by the solution, $f_S(\vec{r},-\vec{v},T)\mathcal{N}\delta(v-v_0)$, of the Lorentz–Boltzmann equation with homogeneous initial condition $f_S(\vec{r},-\vec{v},0)=1$. Here $\mathcal{N}$ is a normalization constant given by $\mathcal{N}=(2\pi v_0)^{-1}$ and $(4\pi v_0^2)^{-1}$ in two and three dimensions, respectively. That is, we need the long time solution of the Lorentz–Boltzmann equation, with absorbing boundary conditions, and with an initial condition which is uniform in space and in velocity directions. Such a solution will not stay uniform due to the escape of particles through the absorbing boundaries.

Therefore Eq. (9) is equivalent to

$$\langle g(\boldsymbol{\rho})\rangle_{\text{Rep}} = \lim_{t\to\infty}\lim_{T\to\infty}\frac{\int d\boldsymbol{\rho}d\vec{r}d\vec{v}F(\vec{r},\vec{v},\boldsymbol{\rho},t)f_S(\vec{r},-\vec{v},T)g(\boldsymbol{\rho})}{\int d\boldsymbol{\rho}d\vec{r}d\vec{v}F(\vec{r},\vec{v},\boldsymbol{\rho},t)f_S(\vec{r},-\vec{v},T)}. \quad (11)$$

Equation (11) reduces the calculation of the sum of the Lyapunov exponents and the maximum Lyapunov exponent on the repeller to standard integrations, when the solution of the Lorentz–Boltzmann equation for $f_S(\vec{r},-\vec{v},T)$ and that of the ELBE for F are obtained. In the following sections we will solve these equations for large systems with absorbing boundary conditions. For long times, t, the solution of the ELBE for F will be obtained as a generalized Chapman–Enskog [16] expansion of the form

$$\begin{aligned}F(\vec{r},\vec{v},\boldsymbol{\rho},t)=&\mathcal{N}\delta(v-v_0)(\psi_0(\boldsymbol{\rho})n_m(\vec{r},t)+\psi_1(\boldsymbol{\rho})\vec{v}\cdot\nabla n_m(\vec{r},t)\\&+\psi_2(\boldsymbol{\rho})v^2\nabla^2 n_m(r,t)+\cdots).\end{aligned} \quad (12)$$

Also $n_m(\vec{r},t)$ is the *slowest decaying eigenmode of the diffusion equation* with the given absorbing boundary conditions,

$$n_m(\partial V)=0. \quad (13)$$

In second order in the gradient we have kept only the scalar part $\nabla^2 n_m$. A possible contribution of order ∇^2 to the Lyapunov exponents from a term in Eq. (12) proportional to the traceless tensor $\mathbf{vv}-(v^2/d)\mathbf{1}$ vanishes after integration with respect to the velocity and is therefore neglected in this equation. The solution of the usual Lorentz–Boltzmann equation for $f_S(\vec{r},-\vec{v},T)$ can also be written as a Chapman–Enskog expansion

$$f_S(\vec{r},-\vec{v},T)=\mathcal{N}\delta(v-v_0)(n_m(\vec{r},T)-c_d\vec{v}\cdot\nabla n_m(\vec{r},T)+\cdots). \quad (14)$$

The constants in Eq. (14) are $c_2=-3/(4\nu)$ and $c_3=-1/\nu$ in two and three dimensions respectively, where ν is the mean collision frequency. Due to the normalization condition, Eq. (6), the functions ψ_i in Eq. (12) have to fulfill the conditions

$$\int d\boldsymbol{\rho}'\,\psi_0(\boldsymbol{\rho}')=1, \quad (15)$$

$$\int d\boldsymbol{\rho}'\,\psi_1(\boldsymbol{\rho}')=c_d, \quad (16)$$

$$\int d\boldsymbol{\rho}'\,\psi_2(\boldsymbol{\rho}')=0. \quad (17)$$

It is important to note that in Eqs. (12) and (14), the eigenmodes $n_m(\vec{r},t)$ and $n_m(\vec{r},T)$ have the functional forms $n_0(\vec{r})\exp[-t\omega]$ and $n_0(\vec{r})\exp[-T\omega]$, respectively, where ω denotes the eigenvalue of the slowest decaying mode of the diffusion equation, and $n_0(\vec{r})$ is the corresponding eigenfunction.

Using Eqs. (12–17) we can write Eq. (11) as

$$\langle g(\boldsymbol{\rho})\rangle_{\mathrm{Rep}}=\frac{\int d\boldsymbol{\rho}\int d^dr\left[\psi_0 n_0^2(\vec{r})-v_0^2\left(\frac{c_d}{d}\psi_1+\psi_2\right)(\nabla n_0(\vec{r}))^2\right]g(\boldsymbol{\rho})}{\int d^dr\left(n_0^2(\vec{r})-\frac{v_0^2c_d^2}{d}(\nabla n_0(\vec{r}))^2\right)}+\cdots. \quad (18)$$

We note that exponentially decaying factors have canceled in the numerator and denominator of Eq. (18).

Since we only kept terms up to the order ∇^2 when deriving Eq. (18), we also have to expand the denominator in Eq. (18). The final result for averaging a quantity $g(\boldsymbol{\rho})$ on the repeller, up to and including terms of order ∇^2, is given by

$$\langle g(\boldsymbol{\rho})\rangle_{\mathrm{Rep}}=\int d\boldsymbol{\rho}\left\{\psi_0+\left[\frac{c_d^2}{d}\left(\psi_0-\frac{\psi_1}{c_d}\right)-\psi_2\right]v_0^2\overline{q^2}\right\}g(\boldsymbol{\rho})+\cdots, \quad (19)$$

with

$$\overline{q^2}=\frac{\int d^dr(\nabla n_0(\vec{r}))^2}{\int d^dr n_0^2(\vec{r})}. \quad (20)$$

The quantity $\overline{q}$ may be interpreted as a wave vector. If, for example, a system is considered with absorbing boundaries at $x=\pm L/2$, but of infinite extent in other directions, Eq. (20) can be evaluated to show that $\overline{q}\equiv q_x^{\min}=(\pi/L)$ is the smallest possible wave vector in the x direction. We will consistently neglect effects due to microscopic, kinetic boundary layers near ∂V, since such effects are unimportant in the escape-rate formalism as the size of the system becomes large.

That this is the case can be understood in the following way: The fraction of the volume taken up by the boundary layers is of order λ/L with λ the mean free path and L a length of the order of the diameter of the system. The density in the boundary layer is equally of order λ/L compared to the average density in the system, as a result of the absorbing boundary condition. Finally the escape rate near the boundary is of similar order, as it is proportional to the local density. As a result the effects of the boundary layer are of order $(\lambda/L)^3$, whereas here we will be interested in terms of order $(\lambda/L)^2$ only.

III. LYAPUNOV EXPONENTS

The strategy presented so far is applicable to the calculation of all quantities which can be written as ensemble averages of functions of the ROC matrix. In our case this always happens, if a quantity is given as a time average over a function of $\boldsymbol{\rho}(t)$. The sum of the positive Lyapunov exponents fulfills this property as shown by Sinai [17]. We *assume* that trajectories on the repeller for the Lorentz gas are sufficiently ergodic, so that we can write the sum of the exponents as an average over the appropriate ensemble of ROC matrices (see also I):

$$\sum_{\lambda_i>0}\lambda_i(\mathcal{R})=v\langle\mathrm{Trace}(\boldsymbol{\rho}^{-1})\rangle_{\mathrm{Rep}}. \quad (21)$$

In two dimensions this is trivially also the largest exponent. The maximum Lyapunov exponent in 3 dimensions is not calculated as a time average, but as an average over a function of time of free flight and collision parameters. The separation of trajectories at a given time can then be written as product of matrices, each describing the propagation of the ROC between two collisions, as explained in I [1]. Using an identical method, we find

$$\delta\vec{r}^{\perp}(t)=\prod_{i=1}^{N}\boldsymbol{U}_i(\tau_i,\phi_i,\alpha_i)\,\delta\vec{r}^{\perp}(0),$$

with

$$\boldsymbol{U}_i(\tau_i,\phi_i,\alpha_i)=T\exp\left(\int_{t_i}^{t_{i+1}} v\boldsymbol{\rho}^{-1}(t',\phi_i,\alpha_i)dt'\right). \tag{22}$$

Here ϕ_i and α_i are the scattering angles at the ith collision and $\tau_i=t_{i+1}-t_i$ is the time of free flight between the ith and $(i+1)$th collision. It is important to notice that Eq. (22) cannot be written in terms of a time integral over a local function of $\boldsymbol{\rho}(t)$, since the $\mathbf{U_i}$ matrices typically do not commute with each other.

Since for dilute Lorentz gases correlations between collision events are not important in the limit of low densities, we can make the approximation that the matrices $\boldsymbol{U}_i$ are independent, randomly, and to the order in gradients that we use here, isotropically distributed matrices, each with independent free flight times and collision parameters selected from appropriate distributions, to be discussed below [18]. We will postpone a discussion of the isotropy of the $\mathbf{U}$ matrices until Sec. V and Appendix B, where we will calculate the largest Lyapunov exponent for three dimensional systems. The maximum Lyapunov exponent is then given by

$$\lambda_{\max}(\mathcal{R})=\lim_{N\to\infty}\frac{1}{\sum_{i=1}^{N}\tau_i}\sum_{i=1}^{N}\ln(|\boldsymbol{U}_i\cdot\vec{e}|), \tag{23}$$

where $\vec{e}$ is an arbitrary unit vector normal to the velocity $\vec{v}$. As in the case of infinite systems, we now write the right-hand side of Eq. (23) as an average over the distribution of the matrices $\boldsymbol{U}_i$. To do this we have first to derive the distribution of particles, which collided a time of free flight τ ago, $f_F(\vec{r},\vec{v},\tau,t)$, for an open system. This is done in Appendix A. Then we have to make sure that the average is restricted to the repeller. This can be achieved as described in Sec. II by using the survival probability. We argue as follows: To determine the appropriate average of $\ln(|\boldsymbol{U}_i\cdot\vec{e}|)$, one needs the distribution of particles that have collided at a point $\vec{r}$, with the scattering angles ϕ,α, that have the velocity $\vec{v}$ after collision, and that travel freely for a time τ until the next collision. Further, to include only trajectories on the repeller, we again need the survival probability, $S(\vec{r},\vec{v},t|t+T)$, for a particle with $\vec{r},\vec{v}$ at time t to remain in the system at least until time $t+T$. We can express the distribution of particles colliding at point $\vec{r}$ and having velocity $\vec{v}$ after collision in terms of the distribution of particles arriving to collide at point $\vec{r}$ with the *restituting* velocity $\vec{v}\,'$, since the rate at which particles arrive at a scatterer before collision should be equal to the rate at which they leave the scatterer after collision. The former rate can be calculated from kinetic theory without difficulty due to the assumption of molecular chaos, while the latter rate requires that correlations between particles and scatterers, produced by a collision, be taken into account. Note also that $|\vec{v}\cdot\hat{n}|=|\vec{v}\,'\cdot\hat{n}|$, and that $\vec{v}\cdot\hat{n}\geqslant 0$, while $\vec{v}\,'\cdot\hat{n}\leqslant 0$. We finally arrive at the following general expression for the maximum Lyapunov exponent of the open system,

$$\lambda_{\max}(\mathcal{R})=\nu_{cr}\langle\ln|\boldsymbol{U}\cdot\vec{e}|\rangle_{cr}=\lim_{T\to\infty}\nu_{cr}\frac{\int d\tau d\hat{n}d\vec{r}d\vec{v}|\hat{n}\cdot\vec{v}|\Theta(\hat{n}\cdot\vec{v})f_F(\vec{r},\vec{v}\,',\tau,t)f_S(\vec{r},-\vec{v},T)\ln|\boldsymbol{U}\cdot\vec{e}|}{\int d\tau d\hat{n}d\vec{r}d\vec{v}|\hat{n}\cdot\vec{v}|\Theta(\hat{n}\cdot\vec{v})f_F(\vec{r},\vec{v}\,',\tau,t)f_S(\vec{r},-\vec{v},T)}. \tag{24}$$

Here $\langle\rangle_{cr}$ is used to distinguish the present average of variables that are calculated only at the instants of collisions from the preceding ones. Further the integrands in both numerator and denominator take into account the effects of particles whose velocities are changed from $\vec{v}\,'$ to $\vec{v}$ at time t, before which a free flight time of τ takes place. Thus, $f_F(\vec{r},\vec{v}\,',\tau,t)$ is the density of particles with velocity $\vec{v}\,'$, before collision, at the point $\vec{r}$ and time t, which had a time of free flight time of length τ before the collision, and ν_{cr} is the average collision frequency for trajectories on the repeller. Here

$$\vec{v}\,'=\vec{v}-2(\vec{v}\cdot\hat{n})\hat{n} \tag{25}$$

and $\hat{n}$ is the unit vector pointing from the center of the sphere (or disk) to the point of impact. The factor $|\hat{n}\cdot\vec{v}|$ appears in the integrals when one takes into account the rate at which collisions take place between the moving particle and the scatterers. Equation (24) should be compared with Eq. (9). The mathematical meanings of these averages are, of course, different. In Eq. (9) the function to be averaged depends on the random matrix $\boldsymbol{\rho}$ and the phase space variables $\vec{r}$ and $\vec{v}$. In Eq. (24), the average depends on the random variables time of free flight τ, collision vector $\hat{n}$ and phase space variables $\vec{r}$ and $\vec{v}$. But in both cases the average is restricted to the repeller.

The distribution of times of free flight $f(\vec{r},\vec{v},\tau,t)$, and therefore the average on the repeller, can be expressed as an expansion in gradients of the density:

$$f_F(\vec{r},\vec{v},\tau,t)=\delta(v-v_0)(\psi_0^F(\tau)n_m(\vec{r},t)+\psi_1^F(\tau)\vec{v}\cdot\nabla n_m(\vec{r},t)+\psi_2^F(\tau)v^2\nabla^2 n_m(r,t)). \tag{26}$$

The functions ψ_i^F are calculated in Appendix A for two and three dimensions. By substituting Eqs. (26) and (14) for f_F and f_S, respectively, one finds that through second order in the gradient, Eq. (24) assumes the form

$$\lambda_{\max}(\mathcal{R})=\nu_{cr}\frac{\int d\tau d\hat{n}|\hat{n}\cdot\vec{v}|\Theta(\hat{n}\cdot\hat{v})\left(\psi_0^F-\left(\frac{c_d}{d}\psi_1^F(1-2(\hat{n}\cdot\hat{v})^2)+\psi_2^F\right)\overline{q}^2v_0^2\right)\ln|\boldsymbol{U}\cdot\hat{e}|}{\int d\hat{n}|\hat{n}\cdot\hat{v}|\Theta(\hat{n}\cdot\hat{v})\left(1-\frac{c_d^2}{d}(1-2(\hat{n}\cdot\hat{v})^2)\overline{q}^2v_0^2\right)}. \tag{27}$$

Here a number of simplifications have already been made. All terms that are odd in the velocity have been left out since they yield zero on integration. Second order terms in the denominator that give zero on integration over the velocity due to the normalization of the Chapman–Enskog solution have been left out likewise. Furthermore terms of the form $\hat{v}\hat{v}:\nabla n\nabla n f(v)$ have been replaced by $(1/d)|\nabla n|^2 f(v)$, which gives the same result on integrating over velocity. Finally the factor $1-2\hat{v}\cdot\hat{n}$ results from integrating $\hat{v}'\cdot\nabla n=(\hat{v}-2(\hat{v}\cdot\hat{n})\hat{n})\cdot\nabla n$ over $\hat{n}$. The components of $\hat{n}$ normal to $\hat{v}$ give vanishing contributions. By defining the constant $A=\int d\hat{n}|\hat{n}\vec{v}|(1-2(\hat{n}\cdot\hat{v})^2)/\int d\hat{n}|\hat{n}\cdot\hat{v}|\Theta(\hat{n}\cdot\hat{v})$, which vanishes in $d=3$ and equals $-1/3$ in $d=2$, and expanding Eq. (27) up to order $\overline{q}^2$, we obtain the final result

$$\lambda_{\max}(\mathcal{R})=\nu_{cr}\int d\tau d\hat{n}|\hat{n}\cdot\hat{v}|\Theta(\hat{n}\cdot\hat{v})\times\left\{\psi_0^F-\left(\psi_2^F(\tau)-\frac{c_d^2}{d}A\psi_0^F(\tau)+\frac{c_d}{d}\psi_1^F(1-2(\hat{n}\cdot\hat{v})^2)\right)\overline{q}^2v_0^2\right\}\times\frac{\ln|\mathbf{U}\cdot\hat{e}|}{\int d\hat{n}|\hat{n}\cdot\hat{v}|\Theta(\hat{n}\cdot\hat{v})}. \tag{28}$$

IV. LYAPUNOV EXPONENT IN TWO DIMENSIONS

There is only one positive Lyapunov exponent in two dimensions. Therefore the positive Lyapunov exponent can either be calculated with the help of Eq. (21), i.e., formally as the sum of the positive Lyapunov exponents or as the maximum Lyapunov exponent with the help of Eq. (24). We will use both methods, to demonstrate that the two approaches lead to the same result. For the two-dimensional Lorentz gas, the radius of curvature matrix reduces to a scalar, ρ, the radius of curvature of two nearby trajectories (see [11] and I), and the ELBE has as variables $\vec{r},\vec{v}$, and ρ. In this case, the ELBE, Eq. (5) has the form [11,1]

$$\frac{\partial F}{\partial t}+\vec{v}\cdot\frac{\partial F}{\partial\vec{r}}+v\frac{\partial}{\partial\rho}F+\nu F=\frac{\nu}{2}\int_{-\pi/2}^{\pi/2}d\phi\cos\phi\,\delta\left(\rho-\frac{a}{2}\cos\phi\right)f_B(\vec{r},\vec{v}',t). \tag{29}$$

Here we have supposed that the typical value of ρ before collision is of the order of the mean free path length, and the delta function appearing in Eq. (5) has been simplified to that appearing in Eq. (29). The solution of the ELBE is normalized according to Eq. (6). With $\vec{v}'$ we denote the velocity of a particle immediately before the collision, which results in a velocity $\vec{v}$ after the scattering event (25). For the solution of Eq. (29) we impose the boundary condition that F vanishes both at zero and at infinite values of ρ, i.e.,

$$F(\vec{r},\vec{v},\rho=0,t)=\lim_{\rho\to\infty}F(\vec{r},\vec{v},\rho,t)=0. \tag{30}$$

We impose the boundary condition at $\rho=0$, because the radius of curvature increases during free motion of the particle, collisions never reduce its value to zero, and all trajectories with an initial negative value for the radius of curvature will acquire positive radius of curvature with probability one. The negative sign of the radius of curvature is conserved only on the stable manifold, which has Lebesgue measure zero in the phase space.

The integration over ϕ can now be performed and we obtain from Eq. (29)

$$\frac{\partial F}{\partial t}+\vec{v}\cdot\frac{\partial F}{\partial\vec{r}}+v\frac{\partial}{\partial\rho}F+\nu F=\frac{\nu}{a}\Theta(1-\sigma)\frac{\sigma}{(1-\sigma^2)^{1/2}}(f_B(\vec{r},\vec{v}'_+,t)+f_B(\vec{r},\vec{v}'_-,t)). \tag{31}$$

where $\sigma=2\rho/a\leqslant 1$. The velocities $\vec{v}\,'_{\pm}$ result from the evaluation of the delta function at precollisional velocities which satisfy the relation $\vec{v}\cdot\hat{n}=\sigma v$. We may use the expansion Eq. (14) for the Lorentz–Boltzmann density $f_B(\vec{r},\vec{v},t)$, by replacing $-\vec{v}$ in Eq. (14) by $\vec{v}$. Then we use the relation $\hat{n}(-\phi)=-\hat{n}(\phi)+2\cos\phi\,\hat{v}$ to obtain, from Eq. (31), an equation for the distribution function $\tilde{F}(\vec{r},\vec{v},\sigma,t)=aF(\vec{r},\vec{v},\rho,t)/2$ given by

$$\frac{\partial\tilde{F}}{\partial t}+\vec{v}\cdot\frac{\partial\tilde{F}}{\partial\vec{r}}+\frac{2v}{a}\frac{\partial}{\partial\sigma}\tilde{F}+\nu\tilde{F}=\nu\Theta(1-\sigma)\frac{\sigma}{(1-\sigma^2)^{1/2}}\times\left(n_m(\vec{r},t)-\frac{3}{4\nu}(1-2\sigma^2)\vec{v}\cdot\nabla n_m(\vec{r},t)\right). \tag{32}$$

Using the Chapman–Enskog *Ansatz*, Eq. (12), for $\tilde{F}$, we obtain equations for ψ_0,ψ_1 and ψ_2 by comparing equal orders of the gradients of the density n_m on both sides of Eq. (32). The zeroth-order equation is ([11] and I)

$$\frac{2v}{a}\frac{\partial}{\partial\sigma}\psi_0+\nu\psi_0=\nu\Theta(1-\sigma)\frac{\sigma}{(1-\sigma^2)^{1/2}}, \tag{33}$$

with solution

$$\psi_0(\sigma)=\begin{cases}\dfrac{a\nu}{2v}e^{-\nu\sigma a/(2v)}, & \text{for } \sigma>1\\[2mm] \dfrac{a\nu}{2v}[1-(1-\sigma^2)^{1/2}], & \text{for } \sigma<1.\end{cases} \tag{34}$$

This solution is continuous at $\sigma=1$ and fulfills the normalization condition Eq. (15) up to corrections of relative order $\tilde{n}=na^2$.

By equating the terms to first order in the gradient in the density in Eq. (32), we obtain an equation for ψ_1, given by

$$\psi_0(\sigma)+\frac{2v}{a}\frac{\partial}{\partial\sigma}\psi_1+\nu\psi_1=-\frac{3}{4}\Theta(1-\sigma)\frac{\sigma(1-2\sigma^2)}{(1-\sigma^2)^{1/2}}. \tag{35}$$

Here we used the fact that the time derivative $\partial n_m/\partial t$ is of second order in the gradient—via the diffusion equation. The solution of Eq. (35) satisfies the normalization condition Eq. (16) with $c_2=-3/(4\nu)$. This can easily be seen by integrating both sides of Eq. (35) with respect to σ. The solution of Eq. (35) has to be continuous at $\sigma=1$ and it is given by

$$\psi_1(\sigma)=\begin{cases}-\dfrac{\nu a^2}{4v^2}\sigma e^{-\nu\sigma a/(2v)}+\dfrac{a}{8v}e^{-\nu\sigma a/(2v)}, & \text{for } \sigma>1\\[2mm] \dfrac{a}{8v}[1-(1-\sigma^2)^{1/2}(1+2\sigma^2)], & \text{for } \sigma<1.\end{cases} \tag{36}$$

By comparing the terms in order $\nabla^2 n_m$ in Eq. (32), we obtain the equation for ψ_2 given by

$$\frac{D}{v^2}\psi_0(\sigma)+\frac{1}{2}\psi_1(\sigma)+\nu\psi_2(\sigma)+\frac{2v}{a}\frac{d}{d\sigma}\psi_2(\sigma)=0. \tag{37}$$

Here we used the diffusion equation for the density n_m

$$\frac{\partial}{\partial t}n_m(\vec{r},t)=D\nabla^2 n_m(\vec{r},t), \tag{38}$$

and the Chapman–Enskog solvability condition, Eq. (17). The low density value of the diffusion coefficient D in two dimensions is $D=3v^2/(8\nu)$. For our purposes it is sufficient to write the solution of Eq. (37) in the form

$$\psi_2(\sigma)=-e^{-\nu a\sigma/2v}\int_0^{\sigma}d\sigma'\,e^{\frac{\nu a\sigma'}{2v}}\left(\frac{D}{v^2}\psi_0(\sigma')+\frac{1}{2}\psi_1(\sigma')\right). \tag{39}$$

Once we know ψ_0,ψ_1, and ψ_2 we can obtain averages on the repeller using Eq. (19). The positive Lyapunov exponent is [see Eq. (21)]

$$\lambda^+(\mathcal{R})=\frac{2v_0}{a}\left\langle\frac{1}{\sigma}\right\rangle_{\text{Rep}}, \tag{40}$$

$$\left\langle\frac{1}{\sigma}\right\rangle_{\text{Rep}}=\int d\sigma\left(\psi_0+\left(\frac{9}{32\nu^2}\left(\psi_0+\frac{4\nu\psi_1}{3}\right)-\psi_2\right)v_0^2\overline{q^2}\right)\frac{1}{\sigma}. \tag{41}$$

Using Eqs. (34), (36), (39) we can now calculate the result in the lowest order in the density by standard integrations, and we obtain, to second order in the gradients,

$$\lambda^+(\mathcal{R})=\lambda_0^+ +\left(\frac{\lambda_0^+}{\nu}-\frac{1}{2}\right)D\overline{q^2}. \tag{42}$$

Here λ_0^+ is the equilibrium solution for the Lyapunov exponent of an infinite system ([11], I) given by

$$\lambda_0^+=2nav[-\ln(2na^2)+1-\mathcal{C}], \tag{43}$$

where $\mathcal{C}$ is Euler's constant.

We can also use the method for the calculation of the largest Lyapunov exponent to obtain the result for λ^+ as an average over the distribution of time of free flight on the repeller. The rank of the matrix $\boldsymbol{U}$, defined in Eq. (22) is unity in two dimensions, i.e, U is a scalar given by

$$U=\left(1+\frac{2v\tau}{a\cos\phi}\right). \tag{44}$$

The leading contribution in the low density approximation is obtained by keeping only the term proportional to τ in Eq. (44). The maximum Lyapunov exponent is then in leading order

$$\lambda^{+}(\mathcal{R})=\frac{1}{\langle\tau\rangle_{cr}}\left\langle \ln\left(\frac{2v}{a}\tau\right)-\ln\cos\phi\right\rangle_{cr}. \tag{45}$$

By using Eqs. (26), (28), and (A3) for the distribution of times of free flight in two dimensions, we can easily recover the result, Eq. (42).

V. LYAPUNOV EXPONENTS IN THREE DIMENSIONS

The ELBE for the three-dimensional Lorentz gas can be solved by choosing an appropriate parameterization of the (ROC) matrix $\boldsymbol{\rho}$. Starting from Eq. (5) we can simplify the delta function appearing in the restituting part of the collision operator (see I), so as to obtain

$$\begin{aligned}\frac{\partial F}{\partial t}&+\vec{v}\cdot\frac{\partial F}{\partial\vec{r}}+v\left(\frac{\partial}{\partial\rho_{11}}+\frac{\partial}{\partial\rho_{22}}\right)F\\&=-\nu F+\frac{\nu}{\pi}\int_0^{\pi/2}d\phi\int_0^{2\pi}d\alpha\,\sin\phi\cos\phi\\&\quad\times\prod_{i\leqslant j}\delta(\rho_{ij}-\rho_{ij}(\phi,\alpha))\\&\quad\times\int d\rho'_{11}\int d\rho'_{12}\int d\rho'_{22}F(\vec{r},\vec{v}\,',\boldsymbol{\rho}',t).\end{aligned} \tag{46}$$

Here the average (infinite system and low density) collision frequency, ν, is $\nu=na^2v\pi$. The ROC matrix is symmetric, therefore only three parameters are necessary to describe its behavior. As shown in I it is convenient to use the eigenvalues ρ_1,ρ_2 and the off diagonal element ρ_{12} as parameters. Together with the normalization condition Eq. (6) and the solution of the Lorentz–Boltzmann equation as a gradient expansion, given by Eq. (14), with $-\vec{v}$ replaced by $\vec{v}$, and $c_3=-1/\nu$,

$$f_B(\vec{r},\vec{v},t)=\mathcal{N}\delta(v-v_0)\left(n_m(\vec{r},t)-\frac{1}{\nu}\vec{v}\cdot\nabla n_m(\vec{r},t)+\cdots\right), \tag{47}$$

with $\mathcal{N}$ defined below Eq. (8). We find that Eq. (46) can be written as

$$\begin{aligned}\frac{\partial\tilde{F}}{\partial t}&+\vec{v}\cdot\frac{\partial\tilde{F}}{\partial\vec{r}}+v\left(\frac{\partial}{\partial\rho_1}+\frac{\partial}{\partial\rho_2}\right)\tilde{F}+\nu\tilde{F}\\&=\frac{\nu}{\pi}\int_0^{\pi/2}d\phi\int_0^{2\pi}d\alpha\,\sin\phi\cos\phi|\cos 2\alpha|\\&\quad\times\delta\left(\rho_1-\frac{a}{2}\cos\phi\right)\delta\left(\rho_2-\frac{a}{2\cos\phi}\right)\\&\quad\times\delta(\rho_{12}+(a\cos\phi\tan^2\phi\sin 2\alpha)/4)f_B(\vec{r},\vec{v}\,',t).\end{aligned} \tag{48}$$

The factor $|\cos 2\alpha|$ results from the transformation from the variables $\rho_{11},\rho_{22},\rho_{12}$ to ρ_1,ρ_2,ρ_{12} in the δ-functions in the right-hand side of Eq. (46), and we set $\tilde{F}(\rho_1,\rho_2,\rho_{12})=F(\rho_{11},\rho_{22},\rho_{12})$. Please note, that $\tilde{F}$ is not yet normalized with respect to ρ_1,ρ_2,ρ_{12}. The velocity $\vec{v}\,'$ before the collision depends on the collision angles α and ϕ, through the relation

$$\vec{v}\,'=\vec{v}-2(\hat{n}\cdot\vec{v})\hat{n},$$

with

$$\hat{n}=\cos\phi\,\hat{v}+\sin\phi\cos\alpha\,\hat{v}_{\perp,1}+\sin\phi\sin\alpha\,\hat{v}_{\perp,2}, \tag{49}$$

and the unit vectors $\hat{v},\hat{v}_{\perp,1},\hat{v}_{\perp,2}$ form an ortho-normal set.

Now, using Eq. (47), we can perform the α integration, and we keep only the non-vanishing terms in the ϕ integration. After integrating over α and making f_B explicit, we find that we can rewrite Eq. (48) as

$$\begin{aligned}&\left[\frac{\partial\tilde{F}}{\partial t}+\vec{v}\cdot\frac{\partial\tilde{F}}{\partial\vec{r}}+v\left(\frac{\partial}{\partial\rho_1}+\frac{\partial}{\partial\rho_2}\right)\tilde{F}+\nu\tilde{F}\right]\\&=\frac{8\nu\mathcal{N}}{\pi a}\int_0^{\pi/2}d\phi\sin\phi\cot^2\phi\\&\quad\times\Theta\left(1-\left|\frac{4\rho_{12}}{a\cos\phi\tan^2\phi}\right|\right)\delta\left(\rho_1-\frac{a\cos\phi}{2}\right)\\&\quad\times\delta\left(\rho_2-\frac{a}{2\cos\phi}\right)\delta(v-v_0)\left(n_m(\vec{r},t)-\frac{1}{\nu}\right.\\&\quad\left.\times(1-2\cos^2\phi)\vec{v}\cdot\nabla n_m(\vec{r},t)+\cdots\right).\end{aligned} \tag{50}$$

Here we have explicitly indicated only the terms needed for our further calculations. It is now convenient to introduce the definitions $2\rho_i/a=\sigma_i$ and $2\rho_{12}/a=\sigma_{12}$, and to perform the ϕ integration. We find that

$$\left[\frac{\partial\tilde{F}}{\partial t}+\vec{v}\cdot\frac{\partial\tilde{F}}{\partial\vec{r}}+2v/a\left(\frac{\partial}{\partial\sigma_1}+\frac{\partial}{\partial\sigma_2}\right)\tilde{F}+\nu\tilde{F}\right]$$
$$=\frac{32\nu\mathcal{N}}{\pi a^3}\Theta\left(1-\left|\frac{2\sigma_1\sigma_{12}}{1-\sigma_1^2}\right|\right)\Theta(1-\sigma_1)\frac{\sigma_1^2}{1-\sigma_1^2}$$
$$\times\delta\left(\sigma_2-\frac{1}{\sigma_1}\right)\delta(v-v_0)\left(n_m(\vec{r},t)-\frac{1}{\nu}\right.$$
$$\left.\times(1-2\sigma_1^2)\vec{v}\cdot\nabla n_m(\vec{r},t)+\cdots\right). \quad (51)$$

We point out that $\tilde{F}$ is not normalized with respect to σ_1,σ_2, and σ_{12}, since we have not yet introduced the appropriate Jacobian. Equation (51) can be further simplified by introducing a new set of variables $(0\leqslant s, 0\leqslant z\leqslant 1, -\pi/2\leqslant\gamma\leqslant\pi/2)$, defined by the relations

$$\sigma_1=z+s,$$
$$\sigma_2=\frac{1}{z}+s,$$
$$\sigma_{12}=\sin(\gamma)\,\frac{1-z^2}{2z}. \quad (52)$$

The *normalized* distribution function $\tilde{f}(\vec{r},\vec{v},s,z,\gamma,t)=J(s,z,\gamma)\tilde{F}(\vec{r},\vec{v},\rho_1,\rho_2,\rho_{12},t)$ obeys the equation

$$\frac{\partial\tilde{f}}{\partial t}+\tilde{\partial}\tilde{f}+\frac{2v}{a}\,\frac{\partial}{\partial s}\tilde{f}+\nu\tilde{f}$$
$$=\frac{2\nu\mathcal{N}}{\pi}z\Theta(1-z)\Theta(z)\Theta(\pi/2-\gamma)\Theta(\gamma+\pi/2)$$
$$\times\delta(s-0^+)\,\delta(v-v_0)\left(n_m(\vec{r},t)-\frac{1}{\nu}\right.$$
$$\left.\times(1-2z^2)\tilde{\partial}n_m(\vec{r},t)\right), \quad (53)$$

where $\tilde{\partial}=\vec{v}\cdot\nabla$, and $J(s,z,\gamma)$ is the Jacobian given by

$$J(z,s,\gamma)=\left|\frac{\partial(\rho_{11},\rho_{22},\rho_{12})}{\partial(\rho_1,\rho_2,\rho_{12})}\right|\left|\frac{\partial(\rho_1,\rho_2,\rho_{12})}{\partial(\sigma_1,\sigma_2,\sigma_{12})}\right|\left|\frac{\partial(\sigma_1,\sigma_2,\sigma_{12})}{\partial(s,z,\gamma)}\right| \quad (54)$$

$$=\frac{a^3}{16}\,\frac{(1+z^2)(1-z^2)}{z^3}. \quad (55)$$

We point out that this Jacobian is independent of s. It guarantees the proper normalization of $\tilde{f}$ as

$$\int_0^\infty ds\int_0^1 dz\int_{-\pi/2}^{\pi/2}d\gamma\tilde{f}(\vec{r},\vec{v},s,z,\gamma,t)=f_B(\vec{r},\vec{v},t), \quad (56)$$

where f_B is the solution of the standard nonequilibrium Lorentz–Boltzmann equation.

The physical meanings of s, z, and γ become a bit more transparent if we consider Eq. (52) and Eq. (53) in more detail. We begin by noting, that the distribution function, $F(\vec{r},\vec{v},\rho_1,\rho_2,\rho_{12},t)$, for the two eigenvalues and the off diagonal element of the ROC matrix is established through the dynamic process involving intervals of free flight separated by collisions of the moving particles with the scatterers. The time dependence of the ROC matrix can be completely expressed in terms of scattering angles and time of free flight. In (I) it was shown that the eigenvalues of the ROC matrix increase linearly in time during a free flight and the off diagonal element stays constant. The reparametrization Eq. (52) exactly reflects this behavior. We can see that the dimensionless parameter s corresponds to the time of flight, which is also clear since the δ - function in s in the right-hand side of Eq. (53) shows that the gain term is a source of particles with a free flight time of zero. From Eq. (52) we can then conclude that z, respectively $1/z$, correspond to the possible values of normalized eigenvalues σ_1, σ_2 at the collision. Equation (48) shows that z is physically the cosine of the scattering angle ϕ. With the same arguments, it can be seen that up to a factor of 2, γ finds its physical correspondence in the azimuthal scattering angle α.

As one might suspect here, the solution of Eq. (53) can be interpreted as a joint distribution function for time of free flight and collision parameter. This may seem surprising at first, because the latter are statistically independent quantities. Correlations in $\tilde{f}$ however, are the result of considering this function at fixed time and position. In a comoving frame, i.e., cconsidering $\tilde{f}(\vec{r}-(a/2)\sigma\hat{v},\vec{v},s,z,\gamma,t-a\sigma/2v)$, one would find the variables s, z, and γ to be uncorrelated indeed.

The distribution of times of free flight in 3 dimensions Eq. (A1) can be recovered, if we integrate Eq. (53) over z and γ and identify $as/(2v)$ with τ:

$$f_F(\vec{r},\vec{v},\tau)=\frac{a}{2v}\int_{-\pi/2}^{\pi/2}d\gamma\int_0^1 dz\tilde{f}\left(\vec{r},\vec{v},\frac{2v}{a}\tau,z,\gamma\right). \quad (57)$$

For large systems, we can solve Eq. (53), as in the 2 dimensional case, by using a gradient expansion

$$\tilde{f}=\delta(v-v_0)\mathcal{N}\Theta(\pi/2-\gamma)\Theta(\gamma+\pi/2)\left(n_m(\vec{r},t)\tilde{\psi}_0(s,z)\right.$$
$$\left.+\frac{a}{2v}\tilde{\partial}n_m(\vec{r},t)\tilde{\psi}_1(s,z)+\frac{a^2}{4}\nabla^2 n_m(\vec{r},t)\tilde{\psi}_2(s,z)+\cdots\right). \quad (58)$$

The solution of Eq. (53) has to satisfy the relation (56) The quantity $\tilde{\psi}_0(s,z)$ was already obtained in I. It satisfies the equation

$$\frac{d}{ds}\tilde{\psi}_0+\tilde{\nu}\tilde{\psi}_0=\frac{2\tilde{\nu}}{\pi}\Theta(1-z)\Theta(z)z\delta(s-0^+), \quad (59)$$

where the dimensionless collision frequency is $\tilde{\nu}=(a/2v)\nu$. The solution of Eq. (59) is

$$\tilde{\psi}_0(s,z)=\frac{2\tilde{\nu}}{\pi}\Theta(1-z)\Theta(z)z\Theta(s)e^{-\tilde{\nu}s}. \qquad (60)$$

The equation for $\tilde{\psi}_1$ is obtained by keeping only the terms proportional to $\tilde{\partial}n_m$

$$\tilde{\psi}_0+\frac{d}{ds}\tilde{\psi}_1+\tilde{\nu}\tilde{\psi}_1=\frac{-2}{\pi}\Theta(1-z)\Theta(z)z(1-2z^2)\delta(s-0^+). \qquad (61)$$

This equation is easily solved to give

$$\tilde{\psi}_1(s,z)=\frac{-2\tilde{\nu}}{\pi}\left(sz+\frac{1}{\tilde{\nu}}z(1-2z^2)\right)\times\Theta(1-z)\Theta(z)\Theta(s)e^{-\tilde{\nu}s}. \qquad (62)$$

Considering the terms of order $\nabla^2 n_m(\vec{r},t)$ and keeping only the scalar part of $\tilde{\partial}\tilde{\partial}$, we obtain the equation for $\tilde{\psi}_2$

$$\hat{D}\tilde{\psi}_0+\frac{1}{3}\tilde{\psi}_1+\frac{d}{ds}\tilde{\psi}_2+\tilde{\nu}\tilde{\psi}_2=0. \qquad (63)$$

Here the Chapman–Enskog solvability condition Eq. (17) and the diffusion equation for n_m, Eq. (38) were used. We also introduced the dimensionless diffusion coefficient $\hat{D}=(2/av)D$. The solution of this equation is given by

$$\tilde{\psi}_2(s,z)=\frac{2\tilde{\nu}}{\pi}\left(-\hat{D}sz+\frac{1}{6}s^2z+\frac{1}{3\tilde{\nu}}sz(1-2z^2)\right)\times\Theta(1-z)\Theta(z)\Theta(s)e^{-\tilde{\nu}s}. \qquad (64)$$

The formulas for the sum of the Lyapunov exponents and the maximum Lyapunov exponent in equilibrium were obtained in I. The same formulas are valid in the nonequilibrium case, if the averages are replaced by averages on the repeller. For the sum of the Lyapunov exponents Eq. (19) gives

$$\frac{a}{2v}\sum_{\lambda_i>0}\lambda_i(\mathcal{R})=\left\langle\left(\frac{1}{z+s}+\frac{1}{1/z+s}\right)\right\rangle_{\text{Rep}}, \qquad (65)$$

while Eq. (28) leads to

$$\lambda_{\max}(\mathcal{R})=\nu_{cr}\langle\ln|\mathbf{U}(\cos\phi,\alpha,\tau)\cdot\vec{e}_\psi|\rangle_{cr}, \qquad (66)$$

with the average collision frequency on the repeller $\nu_{cr}=1/\langle\tau\rangle_{cr}$ and

$$|\mathbf{U}(z,\alpha,\tau)\cdot\vec{e}_\psi|=\frac{2v\tau}{a}\sqrt{\frac{1+z^4+(1-z^4)\cos 2(\psi-\alpha)}{2z^2}} \qquad (67)$$

with $z=\cos\phi$, as derived in I. The angle ψ specifies the direction of the unit vector $\vec{e}_\psi$ in the plane perpendicular to the trajectory. In general an additional average over its stationary distribution $P(\psi)$ is necessary. However, it can easily be shown that the corrections to an isotropic distribution of directions $P(\psi)$ are at most of order $\cos\psi\nabla^2 n(r,t)$ and do not contribute to the average in Eq. (66) in order $\nabla^2 n(r,t)$ (see Appendix B).

Now, using Eqs. (65), (19), we are led to the determination of the sum of the positive Lyapunov exponents, and we obtain, to second order in the gradients,

$$\lambda^+_{\max}(\mathcal{R})+\lambda^+_{\min}(\mathcal{R})=h^0_{KS}-D\overline{q^2}(1+2(\ln(\tilde{n}/2)+\mathcal{C})), \qquad (68)$$

with $\tilde{n}=n\pi a^3=2\tilde{\nu}$. Here, h^0_{KS} is the KS-entropy for an infinite system [I] at equilibrium given by (69), $\mathcal{C}$ is Euler's constant, and $\overline{q}$ is defined by (20), and

$$h^0_{KS}=2na^2v\pi[-\ln(\tilde{n}/2)-\mathcal{C}]+\cdots. \qquad (69)$$

with the dots indicating higher density corrections.

This expression for the sum of the Lyapunov exponents can also be calculated by averaging $\ln\det\boldsymbol{U}=\text{Trace}\ln\boldsymbol{U}$ over the distribution of times of free flight and the scattering angles Ω with the help of Eq. (28), and by replacing $\ln|\boldsymbol{U}\cdot\vec{e}|$ with $\text{Trace}\ln\boldsymbol{U}$. In leading order for large times of free flight, i.e., small density of scatterers, we obtain

$$\lambda^+_{\max}(\mathcal{R})+\lambda^+_{\min}(\mathcal{R})=\nu_{cr}\langle\text{Trace}\ln(v\boldsymbol{\rho}_+^{-1}\tau)\rangle_{cr}. \qquad (70)$$

The matrix $\boldsymbol{\rho}_+$ is the ROC immediately after a scattering event. It depends in leading order in the density only on the scattering angles ϕ and α and is defined in paper I. Here it is only important to notice that it has eigenvalues $\rho_1=(a/2)\cos\phi$ and $\rho_2=a/(2\cos\phi)$. The trace of the logarithm in Eq. (70) leads therefore to a cancellation of the terms depending on $\ln\cos\phi$, so that we only have to evaluate

$$\lambda^+_{\max}(\mathcal{R})+\lambda^+_{\min}(\mathcal{R})=2\nu_{cr}\left\langle\ln\left(\frac{2v}{a}\tau\right)\right\rangle_{cr}. \qquad (71)$$

This strategy also leads to the result Eq. (68).

With the help of Eqs. (66), (67), (24) and Appendix A the maximum Lyapunov exponent is given, to second order in the gradients, by

$$\lambda^+_{\max}(\mathcal{R})=\lambda^0_{\max}+D\overline{q^2}\left(-\ln(\tilde{n}/2)-\mathcal{C}+\frac{1}{4}-\ln 2\right), \qquad (72)$$

where $\lambda^0_{\max}$ is the equilibrium value of the maximum Lyapunov exponent for an infinite system

$$\lambda^0_{\max}=na^2v\pi[-\ln(\tilde{n}/2)+\ln 2-\tfrac{1}{2}-\mathcal{C}]+\cdots. \qquad (73)$$

The expression for the smallest positive Lyapunov exponent can be obtained, to second order, from Eqs. (68) and (72), as

$$\lambda^{+}_{\min}(\mathcal{R})=\lambda^{0}_{\min}+D\overline{q^2}(-\ln(\tilde{n}/2)-\mathcal{C}-\tfrac{5}{4}+\ln 2)\cdots, \tag{74}$$

with

$$\lambda^{0}_{\min}=na^2 v\pi[-\ln(\tilde{n}/2)-\ln 2+\tfrac{1}{2}-\mathcal{C}]+\cdots. \tag{75}$$

VI. DISCUSSION

We have now calculated, analytically, the spectrum of positive Lyapunov exponents on the repeller for the open, dilute, random Lorentz gas in two and three dimensions. Then using the escape-rate formula we may infer the values of the KS entropies on the repeller as well. We find that the corrections to the equilibrium values of the Lyapunov exponents and KS entropies are of order $1/L^2$ where L is some characteristic size of the open system. We should point out that Gaspard has discussed a reformulation of the escape-rate formula so as to be able to express the diffusion coefficient in terms of the Hausdorff dimension of the fractal repeller [9]. Using his method we can easily see that the dimension of the fractal repeller is slightly less than the embedding dimension (3 for $d=2$, and 5 for $d=3$) by terms of order $1/L^2$. These results are to be expected for the Lorentz gas, since diffusion is normal, and the escape-rate formula should be free of difficulties.

Gaspard and Baras [14] have examined the chaotic properties of the periodic open Lorentz gas at sufficiently high densities that there are no infinite horizons for the moving particle. They used numerical simulations, and obtained independent results for the Lyapunov exponents and for the KS entropy on the repeller, as functions of the system size, L. Then they compared these results with numerical and with approximate analytical values for the diffusion coefficient and found good agreement.

At the present time there are no computer simulations of open, random Lorentz gases, to which our analytic results can be compared, but we may compare our results to intuitive expectations. To do this we will need to note that the average collision frequency, ν_{cr} or equivalently, the mean free time between collisions on the repeller, $\tau_{cr}=1/\nu_{cr}$, differ from the corresponding quantities in an infinite system. In fact τ_{cr} can explicitly be calculated by using an expression analogous to Eq. (28), where $\ln|\mathbf{U}\cdot\hat{e}|$ is replaced by τ and the factor ν_{cr} is dropped. Together with the expressions for $\tilde{\psi}_i^F$ in two and three dimension Eqs. (26), (A3), and (A4) we obtain an expression for the mean free time between collisions on the repeller given by

$$\tau_{cr}\equiv\frac{1}{\nu_{cr}}=\frac{1}{\nu}-\frac{D}{\nu^2}\overline{q^2}. \tag{76}$$

This result shows that the mean free time on the repeller is smaller than that in the infinite system. In addition, its dependence on the escape-rate and equilibrium collision frequency is the same in two and three dimensions. Thus, we see that trajectories on the repeller are constrained to have higher collision frequency than those for the infinite, equilibrium system.

The Lyapunov exponents, i.e., the rates of separation of trajectories, are larger on the repeller than in the infinite system. One might have expected, that due to the restriction of the available phase space volume for a particle on the repeller, the rate of separation might have been smaller on the repeller, but the increased scattering rate counteracts this effect. However, if we compensate for this effect by expressing the Lyapunov exponents in units of the mean free time on the repeller, the terms proportional to $\overline{q^2}$ lead to a decrease of the Lyapunov exponents because on average $\ln(\lambda/a)$ decreases. Using Eqs. (42) and (76) for two dimensions and Eqs. (68), (72), (74), and (76) for three dimensions, the Lyapunov exponents in natural units on the repeller are given by

$$\lambda^{+}(\mathcal{R})\tau_{cr}=\frac{\lambda_0^{+}}{\nu}-\frac{1}{2}\frac{D}{\nu}\overline{q^2}\quad\text{in two dimensions,} \tag{77}$$

and for three dimensions

$$\lambda^{+}_{\max}(\mathcal{R})\tau_{cr}=\frac{\lambda^{0}_{\max}}{\nu}-\frac{D}{\nu}\overline{q^2}\left(\frac{8\ln 2-3}{4}\right),$$

$$\lambda^{+}_{\min}(\mathcal{R})\tau_{cr}=\frac{\lambda^{0}_{\min}}{\nu}-\frac{D}{\nu}\overline{q^2}\left(\frac{7-8\ln 2}{4}\right),$$

$$(\lambda^{+}_{\max}(\mathcal{R})+\lambda^{+}_{\min}(\mathcal{R}))\tau_{cr}=\frac{h^0_{KS}}{\nu}-\frac{D}{\nu}\overline{q^2}. \tag{78}$$

It is also interesting to compare the KS entropy on the repeller with its value in an infinite system. To make this a bit more concrete, we consider a slab geometry, i.e., absorbing walls at $x=\pm L/2$, and

$$\overline{q}=\frac{\pi}{L}. \tag{79}$$

Then with Eqs. (42) and (68), respectively, we obtain

$$h_{KS}(\mathcal{R})=h^0_{KS}+\left(\frac{h^0_{KS}}{\nu}-\frac{3}{2}\right)D(\pi/L)^2\quad\text{in two dimensions,} \tag{80}$$

$$h_{KS}(\mathcal{R})=h^0_{KS}+\left(\frac{h^0_{KS}}{\nu}-2\right)D(\pi/L)^2\quad\text{in three dimensions.} \tag{81}$$

Thus the KS entropy increases above its infinite system value when measured in standard time units. As in the case of the Lyapunov exponent, this trend is only due to the increased scattering rate on the repeller. When measured in natural units on the repeller, the KS entropy has an especially simple form

$$h_{KS}(\mathcal{R})\tau_{cr}=\frac{h^0_{KS}}{\nu}-\frac{3}{2}\frac{D}{\nu}(\pi/L)^2\quad\text{in two dimensions,} \tag{82}$$

$$h_{KS}(\mathcal{R})\tau_{cr}=\frac{h_{KS}^0}{\nu}-2\frac{D}{\nu}(\pi/L)^2 \quad \text{in three dimensions.} \tag{83}$$

We conclude with a number of points:

1. The principal problem with the escape-rate method as an analytical method for computing transport coefficients, apart from the inherent difficulties involved in calculating dynamical quantities, is that as yet we have no analytical methods for calculating the KS entropy on the repeller, independently of the escape-rate formula. At the moment we can only use analytic techniques to calculate the diffusion coefficients and the Lyapunov exponents, leaving the KS entropy as a quantity to be derived from them.

2. It would be valuable to have some results from computer simulations with which to compare the results obtained here.

3. As mentioned earlier, the information and Hausdorff dimension of the fractal repellers in both two and three dimensions are very close to the full phase space dimensions, three, for the Lorentz gas on the plane, and five, for the Lorentz gas in space, but are smaller than these values by terms of order L^{-2}. This is a simple consequence of the application of Kaplan–Yorke type formulas to fractal repellers [15]. For the information dimension we obtain

$$d_I=3-2\frac{\gamma}{\lambda_0^+}+O(1/L^4) \quad \text{for} \quad d=2, \tag{84}$$

$$d_I=5-2\frac{\gamma}{\lambda_{\min}^0}+O(1/L^4) \quad \text{for} \quad d=3, \tag{85}$$

where γ is the escape rate and λ_0^+, $\lambda_{\min}^0$ are the positive Lyapunov exponent, and the smaller of the positive Lyapunov exponents in the infinite system, respectively. Gaspard and Baras have used these fractal dimensions to express the diffusion coefficient in terms of the Hausdorff and information dimensions of the fractal repeller for the two dimensional case [9,14].

4. A problem for further study is to extend the calculations given here to a system of many interacting particles such as gases of hard disks or hard spheres. Some progress in this direction has been made, and it is now possible to get analytic results for the KS entropy and the largest Lyapunov exponents for dilute hard disk or hard sphere gases, in equilibrium, in the thermodynamic limit [19–21]. It would be very interesting to apply the escape-rate formalism to the transport coefficients, such as the shear and bulk viscosities, and thermal conductivity, appropriate for fluid systems, using the method of Ref. [8], and to determine the effects of the fractal repeller on the dynamical quantities.

5. The thermodynamic formalism for hyperbolic chaotic systems provides a very useful method for expressing many of the chaotic properties of both open and closed systems in terms of one quantity, the topological pressure [4]. The use of kinetic theory to evaluate the topological pressure for a dilute random Lorentz gas should certainly be possible, but has not yet been undertaken.

6. In the next paper in this series we will consider the case of a Lorentz gas with a charged moving particle placed in an external electric field as well as in a random array of scatterers. Then a Gaussian thermostat is applied which keeps the kinetic energy of the moving particle fixed. The system eventually reaches a nonequilibrium steady state. We will calculate dynamical properties of the moving particle in this nonequilibrium steady state and compare with the results of computer simulations. A preliminary version of this work has already been published [22,23].

ACKNOWLEDGMENTS

A.L. was partially supported by the SFB 262 of the Deutsche Forschungsgemeinschaft. J.R.D. wishes to acknowledge support from the National Science Foundation under Grant No. NSF PHY-96-00428. H.v.B. acknowledges support by FOM and by the NWO Priority Program Non-Linear Systems, which are financially supported by the ''Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO).''

APPENDIX A: THE DISTRIBUTION OF TIMES OF FREE FLIGHT

It is useful to have an expression for the mean collision frequency or, equivalently, for the mean free time between collisions for trajectories on the repeller. Here we argue that such expressions can be obtained in a simple way from a Lorentz–Boltzmann equation, modified so as to include a new variable τ which is the time since the last collision. For the distribution of particles surviving in the system at time t, with position $\vec{r}$, velocity $\vec{v}$, and with time τ since the last collision we propose the equation

$$\frac{\partial}{\partial t}f_F+\vec{v}\cdot\nabla f_F+\dot{\vec{v}}\cdot\frac{\partial}{\partial\vec{v}}f_F+\frac{\partial}{\partial\tau}f_F=-\nu f_F \tag{A1}$$

with

$$f_F(\vec{r},\vec{v}\,',\tau=0,t)=na^{d-1}v\int d\hat{n}\Theta(-\hat{n}\cdot\vec{v})|\hat{n}\cdot\hat{v}|f_B(\vec{r},\vec{v}\,',t). \tag{A2}$$

Equation (A1) can be derived in a way similar to the heuristic derivation of the usual Lorentz–Boltzmann equation, but a few changes have to be made. Scattering and absorption at the boundary are the only mechanisms which will reduce the number of particles with a certain time of free flight. Scattering will also act as a source of particles but always with a time of free flight $\tau=0$. This effect is taken care of by the initial condition Eq. (A2). The dependence of the distribution of times of free flight on both position and velocity is due to the non-uniformity in these variables of f_B resulting from the absorbing boundary condition in combination with a higher survival rate for particles that collide more frequently (they diffuse more slowly). We note one important difference, which will become crucial, if the effect on external fields is considered. In the Boltzmann equation for the one particle phase space density f_B, which is based on the consideration of the number of particles in a fixed volume element in phase space, the streaming term is derived from $(\partial/\partial q)(\dot{q} f_B(q,t))=0$ with $q=(\vec{r},\vec{v})$. This is due to the conservation of particles in the absence of scattering events. In the derivation of f_F we have to consider the number of particles with a certain time of free flight τ after the last scattering event and count how many of them are still there a time step dt later. We therefore have to use a comoving frame, which means that the streaming part of the equation for f_F has the form $\dot{q}(\partial/\partial q) f_F(q,t)$. A term analogous to $f_B(\partial\dot{q}/\partial q)$ cannot appear in a comoving frame since it counts the difference of ingoing and outgoing particles in a *fixed* phase space volume element.

The solution of this equation can be obtained in two and three dimensions as a gradient expansion [see Eq. 26)]. The solution strategy is completely analogous to that used for solving the ELBE in two and three dimensions. For two dimensions we obtain

$$\tilde{\psi}_0^F(\tau)=\nu e^{-\nu\tau},$$
$$\tilde{\psi}_1^F(\tau)=(\tfrac{1}{4}-\nu\tau)e^{-\nu\tau}, \tag{A3}$$
$$\tilde{\psi}_2^F(\tau)=\frac{1}{\nu}\left(-\frac{1}{2}\nu\tau+\frac{1}{4}(\nu\tau)^2\right)e^{-\nu\tau}.$$

In three dimensions we obtain

$$\tilde{\psi}_0^F(\tau)=\nu e^{-\nu\tau},$$
$$\tilde{\psi}_1^F(\tau)=-\nu\tau e^{-\nu\tau}, \tag{A4}$$
$$\tilde{\psi}_2^F(\tau)=\frac{1}{3\nu}\left(-\nu\tau+\frac{1}{2}(\nu\tau)^2\right)e^{-\nu\tau}.$$

As stated before above Eq. (57) the same results may be obtained by integrating $\tilde{f}$ over the variables z and γ.

APPENDIX B: THE DISTRIBUTION OF EIGENDIRECTIONS $P(\psi)$

In order to calculate the maximum Lyapunov exponent of a product of uncorrelated random 2×2 matrices $\Pi U_i(\tau,\phi,\alpha)$, we must take into account the fact that the matrices $\mathbf{U}_i$ do not in general commute with each other. In order to use standard theorems to calculate the largest eigenvalue of a product of random matrices, we have to determine the distribution $P(\psi)$ of the angle ψ, generated by acting with the random matrix $\mathbf{U}$ on the unit vector $\vec{e}(\psi)=(\cos\psi,\sin\psi)$ [18]. If this distribution is not isotropic in ψ, the proper form of $P(\psi)$ must be determined from the solution of an appropriate Frobenius-Perron equation.

In our case, the Frobenius–Perron equation for this distribution is given by

$$P(\psi)=\int_0^{2\pi} d\psi' P(\psi')\langle\delta(\psi-\psi_1(\psi',\phi,\alpha))\rangle. \tag{B1}$$

The angle $\psi_1(\psi',\phi,\alpha)$ is implicitly defined by

$$\mathbf{U}\cdot\vec{e}_{\psi'}=|\mathbf{U}\cdot\vec{e}_{\psi'}|\begin{pmatrix}\cos\psi_1\\ \sin\psi_1\end{pmatrix}, \tag{B2}$$

where ϕ, α are the scattering angles. The distribution function has to obey the normalization condition

$$\int_0^{2\pi} d\psi P(\psi)=1. \tag{B3}$$

The average in Eq. (B1) $\langle\ \dots\ \rangle$ is in our case the average over the distribution of times of free flight, scattering angles, velocities and space multiplied with the survival probability, as defined in Eq. (24). Since, contrary to the case of motion in an external field [23], the equation of motion for $\mathbf{U}$ is the same for an open system as for an infinite system when there is no external field, we can use here the expression of $|\mathbf{U}\cdot\vec{e}_\psi|$ derived in I. To evaluate the ensemble average in the Eq. (B1), we have to use the gradient expansions for the survival probability and the distribution of times of free flight Eqs. (14), (12). Since the ensemble average involves an average over the velocities, the term proportional to $\vec{v}\cdot\vec{\nabla}n$ and all other nonscalar terms vanish. Therefore the deviation of $P(\psi)$ from its isotropic value, $(2\pi)^{-1}$ can at most be of order $\nabla^2 n$. Then, due to the normalization condition Eq. (B3) this deviation must also be be proportional to $\cos(m\psi)$, where m is a positive integer. Consequently the only additional term of order $\nabla^2 n$ in the expression for the maximum Lyapunov exponent would come from multiplying the zeroth-order term (in the gradient of n) of $\langle\ln|\mathbf{U}\cdot\vec{e}(\psi)|\rangle$ (which is independent of ψ) with the term of order $\nabla^2 n$ in the expression for $P(\psi)$. But this averages to zero due to the factor $\cos(m\psi)$. Thus we may use an isotropic distribution in the angle ψ when calculating the largest eigenvalue of the product of random matrices, and we then obtain Eq. (72).

We can understand this result also in a more intuitive way. This demonstration proceeds in four steps.

(1) An isotropic precollisional distribution of ψ will give rise after scattering to anisotropies of order $-\overline{q^2}$. To understand this one should note that angles ψ and $\psi\pm\pi$ can be identified. This merely amounts to an interchange of reference and tangent trajectory. If the contributions of such angle pairs are averaged, the term linear in $\vec{v}\cdot\nabla n$ cancels. The sole source for the anisotropy in this case is the anisotropy of the survival probability as a function of velocity.

(2) A completely anisotropic precollisional distribution, of form $\delta(\psi-\psi_0)$ say, will give rise to a postcollisional distribution that is isotropic in ψ, up to corrections of order $-\overline{q^2}$. This is a consequence of the isotropic distribution of the azimuthal scattering angle α resulting from the random distribution of scatterers. Again, the anisotropy is due to anisotropy of the survival probability and the linear term in $\vec{v}\cdot\nabla n$ vanishes due to identification of ψ and $\psi\pm\pi$.

(3) If we require that the anisotropic part of the distribu-

tion of ψ cancels on averaging, so does the isotropic part of the distribution just after a collision, that is, in an ensemble average in which we put together all contributions of ensemble members with a collision at an equal position and time.

(4) Now since the precollisional anisotropic distribution itself is of order $\overline{q^2}$ according to (1) (and nothing will be added to that just because of the result emerging here), its contribution to the postcollisional anisotropy has to be of order $\overline{q^4}$.

The postcollisional anisotropies resulting from the isotropic part of the precollisional distribution are fully accounted for by Eq. (24) and so this expression gives correct results through order $\overline{q^2}$.

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